

Are there fundamental limits for observing quantum phenomena from within quantum theory?

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Does there exist a limit for the applicability of quantum theory for objects of large mass or size, or objects whose states are of large complexity or dimension of the Hilbert space? The possible answers range from practical limitations due to decoherence within quantum theory to fundamental limits due to collapse models that modify quantum theory. Here, we suggest the viewpoint that there might be also fundamental limits *without altering the quantum laws*. We first demonstrate that for two quantum spins systems of a given spin length, no violation of local realism can be observed, if the measurements are sufficiently coarse-grained. Then we show that there exists a fundamental limit for the precision of measurements due to (i) the Heisenberg uncertainty relation which has to be applied to the measuring apparatus, (ii) relativistic causality, and (iii) the finiteness of resources in any laboratory including the whole universe. This suggests that there might exist a limit for the size of the systems (dimension of the Hilbert space) above which no violation of local realism can be seen anymore.

Despite the enormous success of quantum physics and its wide range of applications, the region of the whole parameter space over which the validity of quantum physics has been directly tested is still rather modest. In Ref. [1], Leggett argues that, taking for example the length scale, it is commonly claimed that quantum laws are valid down to the Planck scale ($\sim 10^{-35}$ m) and up to the size of the characteristic length scale of the Universe ($\sim 10^{+27}$ m). This results in 62 orders of magnitude, compared to about 25 orders of magnitude over which the theory has been directly tested so far. Notwithstanding recent experimental achievements [2–4] that could demonstrate quantum interference in large systems, it remains an open question: *Are there principal limitations on observing quantum phenomena of objects of large mass or size, or objects whose states are of large complexity or dimension of the Hilbert space?*

Here we suggest a possible affirmative answer to the above question when considering the dimensionality of the Hilbert space. This explanation differs conceptually from decoherence [5, 6] or collapse theories [7, 8]. Fully within quantum theory, our approach puts the emphasis on the *observability* of quantum effects and shows that the necessary measurement accuracy to see such effects in systems of sufficiently large Hilbert space dimension cannot be met because of the *conjunction of quantum physics itself, relativity theory, and the finiteness of resources in any laboratory*.

To illustrate our idea, let us start by investigating the experimental requirements for achieving a violation of local realism for systems of increasing Hilbert space dimension. Such a violation in a Bell experiment [9] is generally accepted as a genuine quantum phenomenon. We consider 2 spin- s particles in a generalized singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^s (-1)^{s-m} |m\rangle_A |-m\rangle_B, \quad (1)$$

where, $|m\rangle_A$ ($|m\rangle_B$) denotes the eigenstates of the spin operator's z -component of Alice (Bob). Measuring spin components on either side, this state allows to violate local realism for arbitrarily large s (dimension $2s+1$) but it is necessary that

the *inaccuracy of the angle settings* of Alice and Bob, $\Delta\theta$, is at most in the order of the inverse spin size: $\Delta\theta \lesssim \frac{1}{s}$. This is the case in the Clauser-Horne-Shimony-Holt-type inequality [10] used in Ref. [11], where the difference between setting angles has to be about $\frac{1}{2s+1}$ as well as in Ref. [12] where the setting angle has to fulfil $0 < \sin\theta \approx \theta < \frac{1}{2s}$.

Meanwhile more efficient inequalities for higher-dimensional systems have been found [13, 14]. So one could argue that for the state (1) there might exist Bell inequalities which do not require such a strict condition as $\Delta\theta \lesssim \frac{1}{s}$. However, it has been shown by Peres that for a resolution which is much worse than the intrinsic quantum uncertainty of a spin coherent state, i.e.

$$\Delta\theta \gg \frac{1}{\sqrt{s}}, \quad (2)$$

all Bell inequalities will necessarily be satisfied for the state (1), since the correlations between outcomes of inaccurate measurements become (*classical*) *correlations between classical spins* [11]. This approach was extended to the time evolution of quantum systems and the concept of *macroscopic realism* as introduced by Leggett and Garg [15]. It was shown that for “classical Hamiltonians” and under the restriction of coarse-grained measurements, an arbitrarily large quantum spin evolves as an ensemble of classical spins following a classical mechanical evolution [16].

We will now extend the result of Peres to *arbitrary states* of two spin- s systems, taking the restriction of coarse-grained measurements where *neighboring spin directions* cannot be distinguished. We first introduce the basic mathematical concepts for the further analysis. The (normalized and positive) Q -distribution [17] of a two-system state $\hat{\rho}_{AB}$ is given by

$$Q_{AB}(\Omega_A, \Omega_B) \equiv \left(\frac{2s+1}{4\pi} \right)^2 \langle \Omega_A, \Omega_B | \hat{\rho}_{AB} | \Omega_A, \Omega_B \rangle \quad (3)$$

with Ω_i the spin direction and $|\Omega_i\rangle$ the spin coherent [18] states for system $i = A, B$. In a coarse-grained spin measurement of system i , the whole unit sphere is decomposed into a number

of mutually disjoint angular regions (“slots”) $\Omega_i^{(k)}$, labeled by k . (The decompositions for A and B need not be the same.) A positive operator valued measurement (POVM) on system i has the elements [19]

$$\hat{P}_i^{(k)} \equiv \frac{2s+1}{4\pi} \iint_{\Omega_i^{(k)}} |\Omega_i\rangle\langle\Omega_i| d^2\Omega_i \quad (4)$$

which correspond to these coarse-grained slots ($\sum_k \hat{P}_i^{(k)} = \mathbb{1}$). The joint probability to find the outcome m for system A and the outcome n for system B is given by $w_{AB}^{(mn)} = \text{Tr}[\hat{\rho}_{AB} \hat{P}_A^{(m)} \hat{P}_B^{(n)}]$ or, equivalently, just via integration over the (positive and normalized) Q -distribution:

$$w_{AB}^{(mn)} = \iint_{\Omega_A^{(m)}} \iint_{\Omega_B^{(n)}} Q_{AB}(\Omega_A, \Omega_B) d^2\Omega_A d^2\Omega_B. \quad (5)$$

(Please note that in general $Q_{AB}(\Omega_A, \Omega_B)$ does not factorize, i.e. it cannot be written as a product $Q_A(\Omega_A)Q_B(\Omega_B)$ of two Q -functions of the individual systems.) Upon measurement, the state $\hat{\rho}_{AB}$ is reduced to $\hat{\rho}_{AB}^{(mn)} = \hat{M}_A^{(m)} \hat{M}_B^{(n)} \hat{\rho}_{AB} \hat{M}_A^{(m)\dagger} \hat{M}_B^{(n)\dagger} / w_{AB}^{(mn)}$, with $\hat{M}_i^{(k)}$ the Kraus operators obeying $\hat{M}_i^{(k)\dagger} \hat{M}_i^{(k)} = \hat{P}_i^{(k)}$. The corresponding Q -distribution of the reduced state is $Q_{AB}^{(mn)}(\Omega_A, \Omega_B) = (\frac{2s+1}{4\pi})^2 \langle \Omega_A, \Omega_B | \hat{\rho}_{AB}^{(mn)} | \Omega_A, \Omega_B \rangle$.

Under the restriction of *sufficiently* coarse-grained measurements where the (polar and azimuthal) angular size of these regions, $\Delta\Theta$, has to be much larger than the inverse square root of the spin length s , $\Delta\Theta \gg 1/\sqrt{s}$, the Q -distribution before measurement is very well approximated by the (weighted) mixture of the Q -distributions of the possible reduced states $\hat{\rho}_{AB}^{(mn)}$ [19, 20]:

$$Q_{AB}(\Omega_A, \Omega_B) \approx \sum_m \sum_n w_{AB}^{(mn)} Q_{AB}^{(mn)}(\Omega_A, \Omega_B). \quad (6)$$

Moreover, this condition holds for *all* possible “setting choices” of decompositions for the angular regions of the systems A and B . We could also decompose the regions into a different set of mutually disjoint regions, denoted by $\tilde{\Omega}_i^{(k')}$ (where the decompositions of A and B need not be the same). Then we would get the similar condition $Q_{AB}(\Omega_A, \Omega_B) \approx \sum_{m'} \sum_{n'} w_{AB}^{(m'n')} \tilde{Q}_{AB}^{(m'n')}(\Omega_A, \Omega_B)$, where the $\tilde{Q}_{AB}^{(m'n')}$ are the Q -functions for the reduced states under decomposition into $\tilde{\Omega}_i^{(k')}$. This means that under sufficiently coarse-grained measurements one can consider all results as stemming from an underlying probability distribution, representing a classical ensemble of spins [16, 19]. In particular, there exists a *joint* (positive and normalized) probability $p^{(mm'nn')} \equiv p(\Omega_A^{(m)}, \tilde{\Omega}_A^{(m')}, \Omega_B^{(n)}, \tilde{\Omega}_B^{(n')})$ for the (potential) values corresponding simultaneously to $\Omega_A^{(m)}$ and $\tilde{\Omega}_A^{(m')}$ for spin A and $\Omega_B^{(n)}$ and $\tilde{\Omega}_B^{(n')}$ for spin B , which is given by the integration over the intersections of the corresponding regions:

$$p^{(mm'nn')} = \iint_{\Omega_A^{(m)} \cap \tilde{\Omega}_A^{(m')}} \iint_{\Omega_B^{(n)} \cap \tilde{\Omega}_B^{(n')}} Q_{AB}(\Omega_A, \Omega_B) d^2\Omega_A d^2\Omega_B. \quad (7)$$

All the above can of course be easily generalized to more than two different compositions for each system and to more

than two systems. Q_{AB} can be understood as providing a probability distribution over local hidden variables (Ω_A, Ω_B) . Under sufficiently coarse-grained spin measurements $\Delta\Theta \gg 1/\sqrt{s}$, no Bell inequality can be violated, as a joint probability distribution exists. Therefore, the criterion for having a chance to see deviations from a fully classical description of the two spins reads

$$\Delta\theta \lesssim \frac{1}{\sqrt{s}}. \quad (8)$$

It is clear that for increasingly large s it is hard to meet this experimental requirement and violate local realism or see non-classical correlations. But in fact, we suggest that there might exist even a fundamental upper limit on s —stemming from the Heisenberg uncertainty principle, relativity theory, and finiteness of resources—up to which, for a given measurement device, one can still see non-classical correlations.

The measurements are done with Stern-Gerlach magnets or similar devices. The angle of a magnet has to be set with an accuracy $\Delta\theta$. The Heisenberg uncertainty implies $\Delta L \Delta\theta \geq \hbar/2$, where ΔL is the intrinsic uncertainty of the angular momentum of the whole magnet and $\hbar \sim 10^{-34}$ Js is the reduced Planck constant. Note that in general the form of the angular momentum uncertainty relation is state-dependent as θ is 2π -periodic and its variance is naturally bounded from above [11, 21]. However, in our case of a well aligned measurement apparatus, θ is sharply peaked with a very small width $\Delta\theta \ll 1$ and the problems associated with the periodicity of the angle variable can be neglected. Therefore, the commutator of the angle and angular momentum operator can be written as

$$[\hat{\theta}, \hat{L}] = i\hbar. \quad (9)$$

In Ref. [22] it was shown that the Planck length is a device independent limit which determines the inaccuracy of any distance measurement. Following these thoughts, we can derive a bound on the angular inaccuracy $\Delta\theta$ from within quantum physics. First assume that the spin enters the inhomogeneous magnetic field of the Stern-Gerlach magnet at time $t = 0$ and leaves the interaction zone at time τ . The Hamiltonian of a *freely rotating* magnet is

$$\hat{H} = \frac{\hat{L}^2}{2I}, \quad (10)$$

where \hat{L} is the angular momentum operator of the magnet and $I \sim MR^2$ its moment of inertia with M and R the mass and characteristic size, respectively. (We often neglect factors of the order of 1 throughout our derivations.) In the Heisenberg picture, the time evolution of the polar angle is given by the Heisenberg equation of motion $d\hat{\theta}/dt = -i[\hat{\theta}, \hat{H}]/\hbar = \hat{L}/I$. Therefore, for the measurement duration τ , $\hat{\theta}(\tau) = \hat{\theta}(0) + \hat{L}\tau/I$, where \hat{L} is independent of time. We recall the Robertson inequality $\Delta A \Delta B \geq \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|$ [23], which holds for any two observables \hat{A} and \hat{B} . Using the commutation relation (9), we

obtain

$$\Delta\theta(0) \Delta\theta(\tau) \geq \frac{\hbar\tau}{2I} \sim \frac{\hbar\tau}{MR^2}. \quad (11)$$

It follows that at least *one of the two* quantities, $\hat{\theta}(0)$ and $\hat{\theta}(\tau)$, has a spread of

$$\Delta\theta \gtrsim \frac{1}{R} \sqrt{\frac{\hbar\tau}{M}}, \quad (12)$$

which is denoted as the *standard quantum limit*.

Using condition (8), we obtain the constraint on the spin size such that non-classicality can possibly be seen:

$$s \lesssim \frac{MR^2}{\hbar\tau}. \quad (13)$$

Choosing typical laboratory values $R \sim 1$ m, $M \sim 1$ kg, $\tau \sim 1$ s, one arrives at $s \lesssim 10^{34}$.

In order to obtain a fundamental limit on s , we follow Ref. [22] and impose physical constraints:

By *relativistic causality* the operative size R of the freely moving measurement device cannot exceed the distance that light can travel during the interaction time τ : $R \leq c\tau$, with $c \sim 10^8$ m/s the speed of light. Note that this effective measurement apparatus not only contains the Stern-Gerlach magnet but also the table on which it is mounted and possibly the whole earth etc. Using this constraint, ineq. (12) becomes

$$\Delta\theta \gtrsim \sqrt{\frac{\hbar}{cMR}}, \quad (14)$$

and ineq. (8) then reads

$$s \lesssim \frac{cMR}{\hbar}. \quad (15)$$

Taking again $R \sim 1$ m, $M \sim 1$ kg, this leads to $s \lesssim 10^{42}$.

As a *fundamental limit* one can choose as size and mass of the device the radius and mass of the observable universe, $R_U \sim 10^{27}$ m and $M_U \sim 10^{53}$ kg, respectively. This leads to the condition

$$s \lesssim \frac{cM_U R_U}{\hbar} \sim 10^{122}. \quad (16)$$

Note that under the stronger accuracy condition $\Delta\theta \lesssim \frac{1}{s}$ for the state (1), the spin size for which the Bell inequalities of Refs. [11, 12] can be violated is only $s \lesssim 10^{61}$. Both limits are exceedingly large. However, insofar as the size and mass of the universe as ultimate resources are finite, there is a fundamental limit on how large the Hilbert space of the systems can be such that one is still able to observe genuine quantum features. Despite the fact that the Hilbert space for a spin 10^{61} (10^{122}) can be formed by only about 200 (400) qubits, the question whether or not these two limits are trivial depends on whether a *physical* spin (measured by a Stern-Gerlach apparatus) of such size can in principle be formed.

In order to additionally *avoid gravitational collapse*, the size of the measurement apparatus must be larger than the Schwarzschild radius corresponding to its mass M : $R \geq 2GM/c^2$, with $G \sim 10^{-10}$ m³kg⁻¹s⁻² the gravitational constant [24]. Using this constraint, ineq. (14) becomes

$$\Delta\theta \gtrsim \frac{l_P}{R}, \quad (17)$$

and ineq. (8) then reads

$$s \lesssim \frac{R^2}{l_P^2}, \quad (18)$$

where $l_P \equiv \sqrt{\hbar G/c^3} \sim 10^{-35}$ m is the Planck length. This limit can intuitively be understood since the inaccuracy in the measurement of an angle (which is the ratio of two distances) of, say, a rod of length R is essentially given by the inaccuracy in the position measurement of its extremal point (given by the Planck length) divided by the length of the rod. The latter, of course, has an uncertainty ΔR itself, but this leads to a negligible higher order effect.

For $R \sim 1$ m, we get $s \lesssim 10^{70}$. As an alternative *fundamental limit* one can again take the size of the universe $R_U \sim 10^{27}$ m, which leads to the condition

$$s \lesssim \frac{R_U^2}{l_P^2} \sim 10^{124}. \quad (19)$$

The limit for s in conditions (16) and (19) being similar, reflects the fact that our observable universe is close to be a black hole.

Several remarks have to be made at this point:

- (i) The fact that the standard quantum limit can be beaten by contractive states [25–27] does not change the validity of inequality (12) as *two subsequent* measurements are still bounded by (11) [22].
- (ii) The assumption of a free time evolution (10) must be justified. One could imagine a large setup with all kinds of fields and rods and clever mechanisms which compensates movements of the magnet within itself. But such a construction terminates at the causal radius.
- (iii) One might argue that only *angle differences* are important in the Bell experiment and not the local angles themselves. This fact, however, cannot be exploited as the two (space-time) measurement regions must be *space-like separated* and no rigid connection can exist.
- (iv) We did not take into account *other inaccuracies*, in particular the ones in position and momentum of the spin particles, in the inhomogeneous magnetic field, the state preparation, the ones during the measurement procedure on the screen after the magnet, and inaccuracies in the reference frames of Alice and Bob themselves. All these components of the experimental setup have

to obey the Heisenberg uncertainty as well and maybe impose a much stricter limit. In this sense, we have derived a very conservative *upper bound* on the maximal spin length (cMR/\hbar or R^2/l_p^2 , respectively) beyond which it is impossible to observe the quantum features of an arbitrary state.

- (v) We note that a violation of Bell's inequality remains possible for arbitrarily large spin size s , if the two parties Alice and Bob can perform arbitrary unitary transformations before their measurements even if the latter are still coarse-grained [19, 28]. Consider, for example, the macroscopically entangled state $(|s\rangle_A |-s\rangle_B + |-s\rangle_A |s\rangle_B)/\sqrt{2}$. For observing a violation of Bell's inequality, it is sufficient that Alice and Bob perform coarse-grained which-hemisphere measurements on their local spin systems, but then it is necessary that they have the ability to produce Schrödinger cat-like states of the form $|\pm\alpha\rangle_A = \cos\alpha|s\rangle_A \pm \sin\alpha|-s\rangle_A$. Such a combination of a “non-classical” transformation and a (“classical”) coarse-grained measurement is effectively not a coarse-grained measurement in which *neighboring spin directions in real configuration space are bunched together*. Such measurement observables are denoted as “unreasonable” [11]. (Reasonable coarse-grained observables correspond to measurements that bunch together those outcomes that are neighboring in real space.)

Conclusion. We demonstrated that a violation of local realism cannot be seen for spins of a certain size because of the Heisenberg uncertainty relation, relativistic causality, (gravitational collapse,) and the finiteness of resources in any laboratory. The view taken by most scientists is that the concepts of physical theories being established due to and verified by experiments are independent of the amount of physical resources needed to carry out these experiments. In stark contrast, Benioff [29] and Davies [30] recently argued that physical laws should not be treated as infinitely precise, immutable mathematical constructs, but must rather respect the finiteness of resources in the universe. This might impose a fundamental limit on the precision of the laws and the specifiability of physical states. We enforce this view by proposing that quantum mechanics itself puts a limit on the possibility to observe quantum phenomena if only a restricted amount of physical resources is available.

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- [1] A. J. Leggett, J. Phys.: Cond. Mat. **14**, R415 (2002).
 - [2] M. Arndt, O. Nairz, J. Voss-Andreae, C. Keller, G. van der Zouw, and A. Zeilinger, Nature **401**, 680 (1999).
 - [3] J. R. Friedman, V. Patel, W. Chen, S. K. Tolpygo, and J. E. Lukens, Nature **406**, 43 (2000).
 - [4] B. Julsgaard, A. Kozhekin, and E. S. Polzik, Nature **413**, 400 (2001).
 - [5] W. H. Zurek, Phys. Today **44**, 36 (1991).
 - [6] W. H. Zurek, Rev. Mod. Phys. **75**, 715 (2003).
 - [7] G. C. Ghirardi, A. Rimini, T. Weber, Phys. Rev. D **34**, 470 (1986).
 - [8] R. Penrose, Phil. Trans. R. Soc. Lond. A **356**, 1927 (1998).
 - [9] J. S. Bell, Physics **1**, 195 (1964).
 - [10] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
 - [11] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, 1995).
 - [12] N. D. Mermin, Phys. Rev. D **22**, 356 (1980).
 - [13] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002).
 - [14] M. Junge, C. Palazuelos, D. Perez-Garcia, I. Villanueva, M. M. Wolf, arXiv:0910.4228v1 [quant-ph].
 - [15] A. J. Leggett and A. Garg, Phys. Rev. Lett. **54**, 857 (1985).
 - [16] J. Kofler and Č. Brukner, Phys. Rev. Lett. **99**, 180403 (2007).
 - [17] G. S. Agarwal, Phys. Rev. A **24**, 2889 (1981); G. S. Agarwal, Phys. Rev. A **47**, 4608 (1993).
 - [18] J. M. Radcliffe, J. Phys. A: Gen. Phys. **4**, 313 (1971); P. W. Atkins and J. C. Dobson, Proc. R. Soc. A **321**, 321 (1971).
 - [19] J. Kofler and Č. Brukner, Phys. Rev. Lett. **101**, 090403 (2008).
 - [20] Independent of the concrete implementation, the POVM elements $\hat{P}_i^{(k)}$ (and the Kraus operators $\hat{M}_i^{(k)}$) in sufficiently coarse-grained measurements ($\Delta\Theta \gg 1/\sqrt{s}$) behave almost as projectors for all coherent spin states $|\Omega_i\rangle$ except for those near a slot border, i.e. $\hat{P}_i^{(k)}|\Omega_i\rangle \approx |\Omega_i\rangle$ for Ω_i inside $\Omega_i^{(k)}$ and $\hat{P}_i^{(k)}|\Omega_i\rangle \approx 0$ for Ω_i outside $\Omega_i^{(k)}$ [19].
 - [21] D. T. Pegg, S. M. Barnett, R. Zambrini, S. Franke-Arnold, and M. Padgett, New J. Phys. **7**, 62 (2005).
 - [22] X. Calmet, M. Graesser, and S. D. H. Hsu, Phys. Rev. Lett. **93**, 211101 (2004).
 - [23] H. P. Robertson, Phys. Rev. **34**, 163 (1929).
 - [24] It is of course conceivable that an observer performs experiments inside a black hole. Such an observer, however, cannot communicate his results to regions outside the black hole.
 - [25] H. P. Yuen, Phys. Rev. Lett. **51**, 719 (1983).
 - [26] C. M. Caves, Phys. Rev. Lett. **54**, 2465 (1985).
 - [27] V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004).
 - [28] W. Son, J. Kofler, M. S. Kim, V. Vedral, and Č. Brukner, Phys. Rev. Lett. **102**, 110404 (2009).
 - [29] P. Benioff, arXiv:quant-ph/0303086v3.
 - [30] P. C. W. Davies, in: *Randomness and Complexity, from Leibniz to Chaitin*, ed. C. S. Calude (World Scientific, Singapore, 2007); electronic version: arXiv:quant-ph/0703041v1.